

# Time Dependent Control Lyapunov Functions and Hybrid Zero Dynamics for Stable Robotic Locomotion

Shishir Kolathaya, Ayonga Hereid and Aaron D. Ames

**Abstract**—Implementing state-based parameterized periodic trajectories on complex robotic systems, e.g., humanoid robots, can lead to instability due to sensor noise exacerbated by dynamic movements. As a means of understanding this phenomenon, and motivated by field testing on the humanoid robot DURUS, this paper presents sufficient conditions for the boundedness of hybrid periodic orbits (i.e., boundedness of walking gaits) for time dependent control Lyapunov functions. In particular, this paper considers virtual constraints that yield hybrid zero dynamics with desired outputs that are a function of time or a state-based phase variable. If the difference between the phase variable and time is bounded, we establish exponential boundedness to the zero dynamics surface. These results are extended to hybrid dynamical systems, establishing exponential boundedness of hybrid periodic orbits, i.e., we show that stable walking can be achieved through time-based implementations of state-based virtual constraints. These results are verified on the bipedal humanoid robot DURUS both in simulation and experimentally; it is demonstrated that a close match between time based tracking and state based tracking can be achieved as long as there is a close match between the time and phase based desired output trajectories.

## I. INTRODUCTION

It was shown in [5] that by using a *rapidly exponentially stable control Lyapunov function (RES-CLF)* coupled with *hybrid zero dynamics (HZD)* [7], [12], [17], a wide class of controllers can be realized that create rapidly exponentially convergent hybrid periodic orbits, i.e., stable walking gaits; this was demonstrated experimentally on the 2D underactuated bipedal walking robot, MABEL [6]. To achieve these results, model inversion was used through a feedback linearizing controller that creates a linear relationship between the input and output dynamics. For this linearized model, an optimal linear control input was applied [6], which was picked in such a way that a set of desired output trajectories are tracked by the actual output trajectories in a *rapidly exponential* fashion. The progression of the desired outputs over time was based on the function of the hip position which evolves in a monotonic fashion. In other words, the desired outputs are a function of a phase variable  $\tau$  that is a linear approximation of time (see [16]). While this methodology has resulted in sustained walking for bipedal robots [4], [18], especially in the context of planar walking, it has been observed that as the complexity of the robot

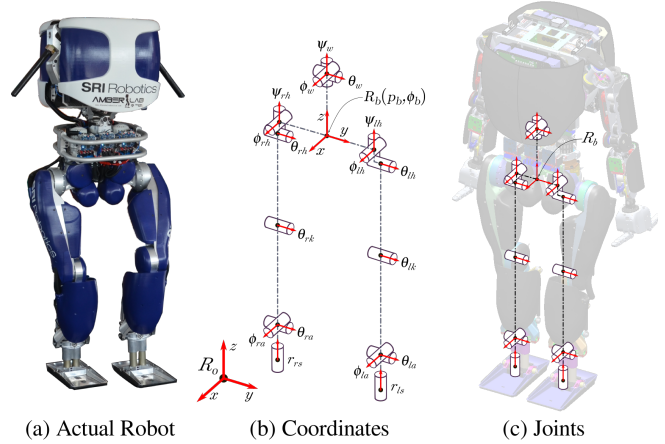


Fig. 1: DURUS robot designed by SRI International.

increases, e.g., 3D humanoid robots, the phase variable can result in a feedback loop between noise in the system and the evolution of the desired output thereby leading to instability.

The humanoid robot DURUS (see Fig. 1), built by SRI International, was demonstrated at the DARPA Robotics Challenge (DRC) Finals during the summer of 2015 (see [2] for a video of the 3D walking displayed). In realizing this dynamic and efficient walking on the humanoid robot, the framework of HZD was utilized. Yet, due to the complexity of the system, utilizing a state-based phase variable  $\tau$  lead to instability due to coupling between high underactuation and sensor noise. Due to stringent scheduling constraints, time-based desired trajectories were implemented on the system with the end result being, unexpectedly, stable locomotion. As a means to understand the time vs. state based implementations formally, the goal of this paper is to study the behavior of time dependent control Lyapunov functions (CLFs) as they relate to their state-based counterparts.

To establish the main result of this paper, by considering the framework of HZD [17] with virtual constraints that are parameterized by a state dependent phase variable,  $\tau$ , we consider the time based virtual constraints (by replacing  $\tau$  with  $t$ ). By considering both time and state based CLFs obtained from these virtual constraints (note that the concept of a time-based CLFs is not a new one, see [11], [10]), we are able to establish ultimate exponential boundedness of the continuous dynamics if the difference between  $t$  and  $\tau$  (and their derivatives) is bounded. These results are extended to the setting of hybrid dynamical systems—which naturally model bipedal walking robots. With the assumption

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that there is an exponentially stable periodic orbit in the hybrid zero dynamics, the main results of this paper are sufficient conditions that ensure exponential boundedness of the periodic orbit for the full order dynamics. That is, we establish (bounded) stability of a walking gait via a time-based CLF under assumptions on the state-dependent phase variable  $\tau$ . Importantly, these results are verified both in simulation and experimentally on DURUS (Fig. 1).

The paper is structured as follows: Section II will introduce the CLF and specifically the RES-CLF for state based outputs which yields rapid convergence to the zero set. This rapid convergence is important in the context of hybrid systems due to the fact that continuous events take a finite time between the discrete transitions. Section III will introduce the *time based RES-CLFs* and will also make the comparison between the state based and the time based controllers derived. In Section IV, it will be shown that exponential boundedness of the controller to a periodic orbit can be achieved and the bound can be explicitly obtained. Finally, in Section V, this will be extended to hybrid systems and boundedness to the hybrid periodic orbit for the full order dynamics will be shown. Section VI will conclude by showing an experimental implementation of the time based tracking controller on the bipedal robot DURUS.

## II. CONTROL LYAPUNOV FUNCTION

The goal of this section is derive Lyapunov functions and time based Lyapunov functions and realize controllers that utilize them for trajectory tracking. We consider affine control systems of the form

$$\begin{aligned}\dot{x} &= f(x, z) + g(x, z)u, \\ \dot{z} &= \Psi(x, z), \\ y &= e(x, z),\end{aligned}\quad (1)$$

where  $x \in X$  is the set of controllable states,  $z \in Z$  is the set of uncontrollable states and  $u \in U$  is the control input.  $f$ ,  $g$ ,  $\Psi$ ,  $e$  and  $u$  are assumed to be locally Lipschitz continuous. In addition, we assume that  $f(0, z) = 0$ , so that the surface  $Z$  is defined by  $x = 0$  with invariant dynamics  $\dot{z} = \Psi(0, z)$ . The dynamics of  $z$  is called the zero dynamics and that of  $x$  is called the transverse dynamics.  $y : X \times Z \rightarrow O$  is the set of outputs. The dimension of the outputs  $y$  is usually the same as the dimension of control input  $u$ .

**Definition 1:** For the system (1), a continuously differentiable function  $V : X \rightarrow \mathbb{R}_+$  is an exponentially stable control Lyapunov function if there exist positive constants  $\underline{c}, \bar{c}, c > 0$  such that for all  $(x, z) \in X \times Z$ .

$$\begin{aligned}\underline{c}\|x\|^2 &\leq V(x) \leq \bar{c}\|x\|^2 \\ \inf_{u \in U} [L_f V(x, z) + L_g V(x, z)u + cV(x)] &\leq 0,\end{aligned}\quad (2)$$

$L_f, L_g$  are the Lie derivatives. We can accordingly define a set of controllers which render exponential convergence of the transverse dynamics:

$$K(x, z) = \{u \in U : L_f V(x, z) + L_g V(x, z)u + cV(x) \leq 0\},$$

which has the control values that result in  $\dot{V} \leq -cV$ .

**Output Tracking.** If a set of actual outputs  $y_a : X \times Z \rightarrow O$  is to track a set of desired trajectories  $y_d : X \times Z \rightarrow O$ , then the objective the controller  $u$  is to drive  $y = y_a - y_d \rightarrow 0$ . If a feedback linearizing controller is used then the input  $u$  is

$$u = (L_g L_f^{l-1}(y_a - y_d))^{-1}(-L_f^l(y_a - y_d) + \mu), \quad (3)$$

where  $l$  is the relative degree of the outputs and  $\mu$  is the linear feedback control law. Applying (3) in (1) results in:  $\ddot{y} = \mu$ , and by choosing a suitable  $\mu$  (see [14]), the objective  $y \rightarrow 0$  can be realized. Note that if a time based trajectory is used for tracking, then the convergence to zero is achieved via a time based feedback linearizing controller.

**State and Time Based Feedback Linearization.** Without loss of generality, we will consider relative degree two systems that model a mechanical system with configuration space  $q \in \mathbb{Q} \subset \mathbb{R}^n$  of size  $n$ , and  $u \in \mathbb{R}^k$  of size  $k$ ,

$$\begin{bmatrix} \ddot{q} \\ \dot{q} \end{bmatrix} = f_q(q, \dot{q}) + g_q(q, \dot{q})u, \quad (4)$$

for  $(q, \dot{q}) \in T\mathbb{Q}$ . Considering actual and desired relative degree two outputs that are functions of  $q$ ,  $y_a : \mathbb{Q} \rightarrow \mathbb{R}^k$ ,  $y_d : \mathbb{Q} \rightarrow \mathbb{R}^k$ , and the desired outputs that are functions of time  $y_d^t : \mathbb{R}_+ \rightarrow \mathbb{R}^k$  and periodic with period  $T > 0$ , we have the state based and time based output representations as

$$\begin{aligned}y(q) &= y_a(q) - y_d(q), \\ y_t(t, q) &= y_a(q) - y_d^t(t).\end{aligned}\quad (5)$$

$y_d(q)$  can also be defined as a function of a phase variable that is periodic,  $\tau : \mathbb{Q} \rightarrow \mathbb{R}$ , and therefore  $y_d(q) = y_d(\tau(q))$ . Walking gaits, viewed as a set of desired periodic trajectories, are often modulated as functions of a phase variable to eliminate the dependence on time [16]. Taking the derivative of (5) twice, we have

$$\ddot{y} = L_f^2(y_a - y_d) + L_g L_f(y_a - y_d)u, \quad (6)$$

$$\ddot{y}_t = L_f^2 y_a + L_g L_f y_a u - \ddot{y}_d^t, \quad (7)$$

for state based and time based outputs respectively. The controllers that linearize the feedback for  $y$ ,  $y_t$  are

$$u = (L_g L_f(y_a - y_d))^{-1}(-L_f^2(y_a - y_d) + \mu), \quad (8)$$

$$u_t = (L_g L_f y_a)^{-1}(-L_f^2 y_a + \ddot{y}_d^t + \mu_t) \quad (9)$$

respectively.  $\mu_t$  is the time based linear feedback law. By defining the vector:  $\eta = [y^T, \dot{y}^T]^T \in \mathbb{R}^{2k}$ , we can reformulate (4) to the form given by (1):

$$\begin{aligned}\dot{\eta} &= \underbrace{\begin{bmatrix} 0_{k \times k} & 1_{k \times k} \\ 0_{k \times k} & 0_{k \times k} \end{bmatrix}}_F \eta + \underbrace{\begin{bmatrix} 0_{k \times k} \\ 1_{k \times k} \end{bmatrix}}_G \mu, \\ \dot{z} &= \Psi(\eta, z).\end{aligned}\quad (10)$$

Similarly, for the time based outputs:  $\eta_t = [y_t^T, \dot{y}_t^T]^T \in \mathbb{R}^{2k}$ , (4) can be reformulated as

$$\dot{\eta}_t = F \eta_t + G \mu_t, \quad \dot{z}_t = \Psi_t(\eta_t, z_t). \quad (11)$$

$z_t$  is the set of states normal to  $\eta_t$  and has the invariant dynamics  $\dot{z}_t = \Psi_t(0, z_t)$ .

In order to drive  $y, y_t \rightarrow 0$ , we can choose  $\mu, \mu_t$  via control Lyapunov functions in the following manner:

$$\begin{aligned} V(\eta) &= \eta^T P \eta, \\ K(\eta) &= \{\mu : L_F V(\eta) + L_G V(\eta) \mu + \gamma V(\eta) \leq 0\}, \end{aligned} \quad (12)$$

for state based outputs, and

$$\begin{aligned} V^t(\eta_t) &= \eta_t^T P \eta_t, \\ K^t(\eta) &= \{\mu_t : L_F V^t(\eta_t) + L_G V^t(\eta_t) \mu_t + \gamma V^t(\eta_t) \leq 0\}, \end{aligned} \quad (13)$$

for time based outputs.  $P$  is the solution to the continuous algebraic Riccati equation (CARE),  $L_G, L_F$  are the Lie derivatives that are explicitly obtained (see [5]) as follows:

$$\begin{aligned} L_F V &= \eta^T (F^T P + P F) \eta + \gamma V, \quad L_G V = 2\eta^T P G, \\ L_F V^t &= \eta_t^T (F^T P + P F) \eta_t + \gamma V^t, \quad L_G V^t = 2\eta_t^T P G. \end{aligned} \quad (14)$$

**RES-CLF.** Since we need stronger bounds of convergences for hybrid systems (bipedal robots), a *rapidly exponentially stable control Lyapunov function (RES-CLF)* is constructed that stabilizes the output dynamics in a rapidly exponential fashion (see [5] for more details). RES-CLFs will be constructed for state based outputs first and then extended to time based outputs in the next section. Choosing  $\varepsilon > 0$ :

$$V_\varepsilon(\eta) := \eta^T \begin{bmatrix} \frac{1}{\varepsilon} I & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} \frac{1}{\varepsilon} I & 0 \\ 0 & I \end{bmatrix} \eta =: \eta^T P_\varepsilon \eta. \quad (15)$$

It can be verified that this is a RES-CLF in [5]. Besides, the bounds on RES-CLF can be given as

$$c_1 \|\eta\|^2 \leq V_\varepsilon(\eta) \leq \frac{c_2}{\varepsilon^2} \|\eta\|^2, \quad (16)$$

where  $c_1, c_2 > 0$  are the minimum and maximum eigenvalues of  $P$ , respectively. Differentiating (15) yields

$$\dot{V}_\varepsilon(\eta) = L_F V_\varepsilon(\eta) + L_G V_\varepsilon(\eta) \mu, \quad (17)$$

where  $L_F V_\varepsilon(\eta) = \eta^T (F^T P_\varepsilon + P_\varepsilon F) \eta$ ,  $L_G V_\varepsilon(\eta) = 2\eta^T P_\varepsilon G$ . We pick  $\mu$  that results in rapid exponential convergence, i.e.,  $\dot{V}_\varepsilon(\eta) \leq -\frac{\gamma}{\varepsilon} V_\varepsilon(\eta)$  implying

$$L_F V_\varepsilon(\eta) + L_G V_\varepsilon(\eta) \mu \leq -\frac{\gamma}{\varepsilon} V_\varepsilon(\eta), \quad (18)$$

where  $\gamma$  with the usual meaning is obtained from CARE. Therefore, we can define a class of controllers

$$K_\varepsilon(\eta) = \{u \in \mathbb{R}^k : L_F V_\varepsilon(\eta) + L_G V_\varepsilon(\eta) u + \frac{\gamma}{\varepsilon} V_\varepsilon(\eta) \leq 0\}, \quad (19)$$

which yields the set of control values that satisfies the desired convergence rate.

### III. TIME DEPENDENT RES-CLF

Since the primary goal of this paper is to show stability of time dependent feedback control laws, for (4), we first define the time dependent RES-CLF and then, similar to (19), design the class of controllers that drives the time dependent outputs  $y_t$  rapidly exponentially to zero. Defining

$$V_\varepsilon^t(\eta_t) := \eta_t^T P_\varepsilon \eta_t, \quad (20)$$

as in (16), the bounds on  $V_\varepsilon^t$  can be similarly defined:

$$c_1 \|\eta_t\|^2 \leq V_\varepsilon^t(\eta_t) \leq \frac{c_2}{\varepsilon^2} \|\eta_t\|^2, \quad (21)$$

where it must be noted that  $P_\varepsilon$  and the bounds  $c_1, c_2$  remain the same. Accordingly, we can define a class of controllers

$$K_\varepsilon^t(\eta_t) = \{u \in \mathbb{R}^k : L_F V_\varepsilon^t(\eta_t) + L_G V_\varepsilon^t(\eta_t) u + \frac{\gamma}{\varepsilon} V_\varepsilon^t(\eta_t) \leq 0\}, \quad (22)$$

which yields the set of control values that satisfies the desired convergence rate for time dependent outputs. A specific example of  $\mu, \mu_t$  that belong to  $K_\varepsilon$  (19) and  $K_\varepsilon^t$  (22), respectively are the PD controllers:

$$\mu^{PD} = -\frac{1}{\varepsilon^2} K_P y - \frac{1}{\varepsilon} K_D \dot{y}, \quad (23)$$

$$\mu_t^{PD} = -\frac{1}{\varepsilon^2} K_P y_t - \frac{1}{\varepsilon} K_D \dot{y}_t, \quad (24)$$

where  $K_P, K_D$  are chosen such that the matrix  $\begin{bmatrix} 0 & I \\ -K_P & -K_D \end{bmatrix}$  is Hurwitz (see [5]). This will be used specifically in the bipedal robot DURUS which is explained more in Section VI.

**State based vs. time based RES-CLFs.** Given the class of controllers  $K_\varepsilon^t(\eta_t)$  that drive the time based outputs  $\eta_t \rightarrow 0$ , it is important to compare the evolution of the state based outputs  $\eta$ . By assumption of Theorem 1 in [5], the class of controllers  $K_\varepsilon$  yields a locally exponentially stable periodic orbit for the continuous dynamics. The primary goal of this Section and Section IV is to establish conditions for boundedness of the same periodic orbit via using the time dependent controllers  $K_\varepsilon^t$ . Therefore picking the input (9) on the dynamics (6), we have

$$\begin{aligned} \ddot{y} &= L_f^2 y + L_g L_f y u_t, \\ \Rightarrow \ddot{y} &= \underbrace{L_f^2 y + L_g L_f y u}_{=\mu} + \underbrace{L_g L_f y (u_t - u)}_{=:d}, \\ \Rightarrow \ddot{y} &= \mu + d, \end{aligned} \quad (25)$$

where  $d = L_g L_f y (L_g L_f y a)^{-1} (-L_f^2 y a + \ddot{y}_d^t + \mu_t) - \mu + L_f^2 y$  is obtained by substituting for  $u_t, u$  (from (8),(9)). The expression for  $d$  can be further simplified to get

$$d(t, q, \dot{q}, \ddot{q}, \mu_t, \mu) = (\mu_t - \mu) + (\ddot{y}_d^t - \ddot{y}_d). \quad (26)$$

If a phase variable is substituted,  $y_d(q) = y_d(\tau(q))$  (for bipedal robots), then it can be observed that  $d$  becomes small by minimizing the error  $\ddot{y}_d^t - \ddot{y}_d$ . Therefore,  $d$  can be termed *time-phase uncertainty*, or just *phase uncertainty*. This motivates establishing boundedness of the state based outputs  $y$  given that  $d$  is bounded. Going back to (25), we can reformulate (4) that results in the following representation:

$$\begin{aligned} \dot{\eta} &= F \eta + G \mu + G d, \\ \dot{z} &= \Psi(\eta, z). \end{aligned} \quad (27)$$

From the point of view of the state dependent outputs  $\eta$ , we have the following representation dynamics of RES-CLF:

$$\dot{V}_\varepsilon = \eta^T (F^T P_\varepsilon + P_\varepsilon F) \eta + 2\eta^T P_\varepsilon G \mu + 2\eta^T P_\varepsilon G d. \quad (28)$$

For the linear feedback law  $\mu(\eta) \in K_\varepsilon(\eta)$  from (19), the following is obtained:

$$\dot{V}_\varepsilon \leq -\frac{\gamma}{\varepsilon} V_\varepsilon + 2\eta^T P_\varepsilon G d, \quad (29)$$

which captures the underlying theme of the paper establishing the relationship between the *phase uncertainty* and the convergence of  $V_\varepsilon$ . It must be noted that even though time dependent RES-CLF ( $V_\varepsilon^t$ ) leads to convergence of time dependent outputs  $y_t \rightarrow 0$ , (29) extends it to state based outputs  $y$  that are driven exponentially to an ultimate bound, and this ultimate exponential bound is explicitly derived from  $d$ . This is discussed in the next section.

#### IV. EXPONENTIAL BOUNDEDNESS AND ZERO DYNAMICS

Given a stable periodic orbit in the zero dynamics, Theorem 1 in [5] shows that using CLF,  $\mu \in K_\varepsilon(\eta)$ , yields a periodic orbit in the full dynamics that is also exponentially stable. By using a time based CLF,  $\mu_t \in K_\varepsilon^t(\eta_t)$ , we cannot guarantee stability of the same periodic orbit. But, it is possible to show exponential boundedness of the full dynamics to the periodic orbit (see [13]) under certain conditions. Therefore, the goal of this section is to show exponential boundedness for continuous dynamics and to compute the resulting bound.

We will first show exponential boundedness of the output dynamics  $\eta$  and then extend it to the zero dynamics to include the entire system. By considering  $\gamma_1 > 0, \gamma_2 > 0$ , which satisfy  $\gamma = \gamma_1 + \gamma_2$ , we can rewrite  $\frac{\gamma}{\varepsilon} V_\varepsilon = \frac{\gamma_1}{\varepsilon} V_\varepsilon + \frac{\gamma_2}{\varepsilon} V_\varepsilon$  in (29). The first term can thus be used to cancel the input  $d$  and yield exponential convergence until  $\|\eta\|$  becomes sufficiently small. The following lemma has the details.

**Lemma 1:** *Given the class of controllers  $\mu_t(\eta_t) \in K_\varepsilon^t(\eta_t)$  and  $\delta > 0, \exists \beta_\eta > 0$  such that whenever  $\|d\| \leq \delta, \dot{V}_\varepsilon(\eta) < -\frac{\gamma_2}{\varepsilon} V_\varepsilon(\eta) \forall V_\varepsilon(\eta) > \beta_\eta$ .*

Describing the bounds in terms of the coordinates  $\eta$ ,

$$\|\eta(t)\| \leq \frac{1}{\varepsilon} \sqrt{\frac{c_2}{c_1}} e^{-\frac{\gamma_2}{2\varepsilon} t} \|\eta(0)\| \quad \text{for } V_\varepsilon(\eta) > \beta_\eta. \quad (30)$$

**Zero Dynamics.** Given boundedness of the Lyapunov function  $V_\varepsilon(\eta)$ , only the outputs are considered bounded to a ball of radius  $\beta_\eta$ . Therefore, boundedness of the entire system needs to be investigated. Assume that there is an exponentially stable periodic orbit in the zero dynamics, denoted by  $\mathcal{O}_z$ . Let  $\|z\|_{\mathcal{O}_z}$  represent the distance between  $z$  and the nearest point on the periodic orbit  $\mathcal{O}_z$  (see [5]). This means that there is a Lyapunov function  $V_z : Z \rightarrow \mathbb{R}_+$  in a neighborhood  $B_r(\mathcal{O}_z)$  of  $\mathcal{O}_z$  (see [8]) such that

$$\begin{aligned} c_3 \|z\|_{\mathcal{O}_z}^2 &\leq V_z(z) \leq c_4 \|z\|_{\mathcal{O}_z}^2, \\ \dot{V}_z(z) &\leq -c_5 \|z\|_{\mathcal{O}_z}^2, \\ \left\| \frac{\partial V_z}{\partial z} \right\| &\leq c_6 \|z\|_{\mathcal{O}_z}. \end{aligned} \quad (31)$$

Define the composite Lyapunov function:  $V_c(\eta, z) = \sigma V_z(z) + V_\varepsilon(\eta)$ , the following theorem shows ultimate exponential

boundedness of the dynamics of the robot when  $\|d\|$  is bounded.

**Theorem 1:** *Let  $\mathcal{O}_z$  be an exponentially stable periodic orbit of the zero dynamics (10). Given the class of controllers  $\mu_t(\eta_t) \in K_\varepsilon^t(\eta_t)$  and  $\delta > 0, \exists \beta_z, \beta_\eta > 0$  such that whenever  $\|d\| \leq \delta, V_c(\eta, z)$  is exponentially convergent for all  $V_c(\eta, z) > \beta_z + \beta_\eta$ .*

We can also use the notion of Input to State Stability (ISS) by stating that the system (27) is Input to State Stable w.r.t. the input  $d$  (see [15]). In fact, Theorem 1 is a restatement of the notion of exponential input to state stability and the resulting bounds are ensured by restricting  $\|d\| \leq \delta$ .

#### V. HYBRID DYNAMICS

We can now extend the exponential boundedness of the full dynamics for hybrid systems (see Fig. 2). We define a hybrid system in the following manner:

$$\mathcal{H} = \begin{cases} \dot{\eta} &= F\eta + G\mu + Gd, \\ \dot{z} &= \Psi(\eta, z), \\ \eta^+ &= \Delta_\eta(\eta^-, z^-), \\ z^+ &= \Delta_z(\eta^-, z^-), \end{cases} \quad \begin{aligned} &\text{if } (\eta, z) \in \mathbb{D} \setminus \mathbb{S}, \\ &\text{if } (\eta^-, z^-) \in \mathbb{S}, \end{aligned}$$

where  $\mathbb{D}, \mathbb{S}$  are the domain and switching surfaces and are given by

$$\mathbb{D} = \{(\eta, z) \in X \times Z : h(\eta, z) \geq 0\}, \quad (32)$$

$$\mathbb{S} = \{(\eta, z) \in X \times Z : h(\eta, z) = 0 \text{ and } \dot{h}(\eta, z) < 0\},$$

for some continuously differentiable function  $h : X \times Z \rightarrow \mathbb{R}$ .  $\Delta(\eta, z) = (\Delta_\eta(\eta, z), \Delta_z(\eta, z))$  is the reset map representing the discrete dynamics of the system. For the bipedal robot, it represents the impact dynamics of the system, where plastic impacts are assumed. We have the following bounds on reset map:

$$\begin{aligned} \|\Delta_\eta(\eta, z) - \Delta_\eta(0, z)\| &\leq L_1 \|\eta\|, \\ \|\Delta_z(\eta, z) - \Delta_z(0, z)\| &\leq L_2 \|\eta\|, \end{aligned} \quad (33)$$

where  $L_1, L_2$  are the Lipschitz constants.

In order to obtain bounds on the output dynamics for hybrid periodic orbits, it is assumed that  $\mathcal{H}$  has a hybrid zero dynamics for state based control law given by (8) and (19). More specifically we assume that  $\Delta_\eta(0, z) = 0$ , so that the surface  $Z$  is invariant under the discrete dynamics. The hybrid zero dynamics can be described as

$$\mathcal{H}|_Z = \begin{cases} \dot{z} &= \Psi(0, z), & \text{if } z \in Z \setminus (\mathbb{S} \cap Z), \\ z^+ &= \Delta_z(0, z^-), & \text{if } z^- \in (\mathbb{S} \cap Z). \end{cases}$$

Let  $\phi_t(\eta, z)$  be the flow of (27) with initial condition  $(\eta, z)$ . The flow  $\phi_t$  is periodic with period  $T > 0$ , and a fixed point  $\phi_T(\eta^*, z^*) = (\eta^*, z^*)$ . Associated with the periodic flow is the periodic orbit  $\mathcal{O} = \{\phi_t(\Delta(\eta^*, z^*)) : 0 \leq t \leq T\}$ . Similarly, we denote the flow of the zero dynamics  $\dot{z} = \Psi(0, z)$  by  $\phi_T|_Z$  and for a periodic flow we denote the corresponding periodic orbit by  $\mathcal{O}_z \subset Z$ . The periodic orbit in  $Z$  corresponds to a periodic orbit for the full order dynamics,  $\mathcal{O} = \iota_0(\mathcal{O}_z)$ , through the canonical embedding,  $\iota_0 : Z \rightarrow X \times Z$ , given by  $\iota_0(z) = (0, z)$ .

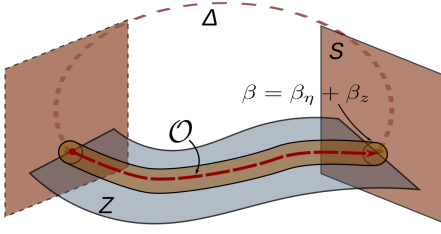


Fig. 2: Figure showing the surface  $Z$ , the hybrid periodic orbit  $\mathcal{O}$ , and the bound  $\beta = \beta_\eta + \beta_z$ .

**Main Theorem.** We can now introduce the main theorem of the paper. Similar to the continuous dynamics, it is assumed that the periodic orbit  $\mathcal{O}_z$  is exponentially stable in the hybrid zero dynamics. Without loss of generality, we can assume that  $\eta^* = 0$ ,  $z^* = 0$ .

**Theorem 2:** Let  $\mathcal{O}_z$  be an exponentially stable periodic orbit of the hybrid zero dynamics  $\mathcal{H}|_Z$  transverse to  $\mathbb{S} \cap Z$ . Given the controller  $\mu_i(\eta_i) \in K_\varepsilon^l(\eta_i)$  for the hybrid system  $\mathcal{H}$ , and given  $r > 0$  such that  $(\eta, z) \in B_r(0, 0)$ ,  $\exists \delta, \bar{\varepsilon} > 0$  such that whenever  $\|d\| < \delta$ ,  $\varepsilon < \bar{\varepsilon}$ , the orbit  $\mathcal{O} = \iota_0(\mathcal{O}_z)$  is ultimately bounded by  $\beta = \beta_\eta + \beta_z$ .

Fig. 2 depicts the periodic orbit  $\mathcal{O}$  and its tube, which is defined by the bound  $\beta$ . Note that the bound  $\beta$  is defined on the Lyapunov function that is a function of a norm of the distance between the state and the periodic orbit  $\mathcal{O}$ . Theorem 2 means that by using a time dependent RES-CLF, any trajectory starting close to the tube will ultimately enter the tube defined by  $\beta$  as long as  $\|d\| < \delta$ .

## VI. SIMULATION AND EXPERIMENTAL RESULTS OF DURUS

DURUS consists of fifteen actuated joints throughout the body and one linear passive spring at the end of each leg. The generalized coordinates of the robot are described in Fig. 1 (see [9]); the continuous dynamics of the bipedal robot is given by the following:

$$D(q)\ddot{q} + H(q, \dot{q}) = Bu + J^T(q)F, \quad (34)$$

$$J(q)\dot{q} + \dot{J}(q, \dot{q})\dot{q} = 0, \quad (35)$$

where  $J(q)$  is the Jacobian of contact constraints and  $F$  is the associated contact wrenches. Due to the presence of passive springs, the double-support domain is no longer trivial. Therefore, a two-domain hybrid system model is utilized to model DURUS walking, where a transition from double-support to single-support domain takes place when the normal force on non-stance foot reaches zero, and a transition from single-support to double support domain occurs when the non-stance foot strikes the ground [9]. Since there is no impact while transitioning from double-support to single-support, the discrete map is an identity. The holonomic constraints are defined in such a way that feet are flat on the ground when they are in contact with the ground. In addition, DURUS is supported by a linear boom which restricts the motion of the robot to the sagittal plane. The contacts with the boom are also modeled as holonomic constraints.

**Outputs and Control.** Considering the planar constraints of the boom, we only consider joint angles that are normal to the sagittal plane, e.g., commanding zero position to all roll and yaw joints. In particular, we define a relative degree one output as

$$y^1(q) = \delta \dot{p}_{hip}(q) - v_d, \quad (36)$$

where  $\delta p_{hip}(q)$  is the linearized hip position,

$$\delta p_{hip}(q) = l_a \theta_{ra} + (l_a + l_c) \theta_{rk} + (l_a + l_c + l_t) \theta_{rh}, \quad (37)$$

with  $l_a$ ,  $l_c$ , and  $l_t$  the length of ankle, calf, and thigh link of the robot respectively.  $v_d$  is a constant desired velocity. The relative degree two outputs are defined in the following (assuming right leg is the stance leg):

- stance knee pitch:  $y_{a,skp}^2 = \theta_{rk}$ ,
- stance torso pitch:  $y_{a,stp}^2 = -\theta_{ra} - \theta_{rk} - \theta_{rh}$ ,
- waist pitch:  $y_{a,wp}^2 = \theta_w$ ,
- non-stance knee pitch:  $y_{a,nskp}^2 = \theta_{lk}$ ,
- non-stance foot pitch:  $y_{a,nsfp}^2 = p_{nst}^z(q) - p_{nsh}^z(q)$ ,
- non-stance slope:

$$y_{a,nsf}^2 = -\theta_{ra} - \theta_{rk} - \theta_{rh} + \frac{l_c}{l_c + l_t} \theta_{lk} + \theta_{lh},$$

where  $p_{nst}^z(q)$  and  $p_{nsh}^z(q)$  are the height of non-stance toe and heel respectively. To guarantee that the non-stance foot remains flat, the desired non-stance foot pitch output should be zero. Correspondingly, the desired outputs  $y_d^2(\tau, \alpha)$  are defined as 6<sup>th</sup>-order B ezier polynomials, where  $\tau$  is the phase variable which can be either a function of time  $t$  directly, or a function of the configuration space  $q$ . The state based phase variable  $\tau(q)$  is defined as  $\tau(q) := \frac{\delta p_{hip}(q) - \delta p_{hip}(q^+)}{v_d}$ . The combined outputs of the system are defined as  $y_a(q) = [y_a^1, y_a^2]^T$  and  $y_d(\tau, \alpha) = [v_d, y_d^2(\tau, \alpha)]^T$ , and the feedback control law in (8) and (9) is applied on the system. The linear feedback laws picked are (23). For the experimental setup, the time based desired outputs:  $y_d^i(t) = y_d(t, \alpha)$ , are picked and PHZD reconstruction applied (see [3]) to obtain the desired configuration angles ( $q_d$ ) and their derivatives ( $\dot{q}_d$ ). A linear feedback law is then applied as the torque input  $u = -\frac{1}{\varepsilon^2} K_P(q - q_d) - \frac{1}{\varepsilon} K_D(\dot{q} - \dot{q}_d)$  to (4). Following results were obtained: Fig. 3 shows the evolution of the outputs over time for both the experiment and simulation, Fig. 4 shows phase portraits, Fig. 5 shows the evolution of the state based and time based phase variable, Fig. 6 shows the walking tiles. The error between the phase variables is low for simulation  $\|d\|_{max} < 9$ . See [1] for the corresponding movie.

## VII. CONCLUSIONS

This paper studied the problem of time dependent CLFs, specifically, those functions that focus on tracking a set of time dependent periodic trajectories. The desired trajectories, typically rendered functions of a phase variable  $\tau$ , are modulated from 0 to the time period  $T$ . A comparison was made between the time based and state based control laws, and it was formally shown that time dependent CLFs can realize state based tracking with acceptable errors as long as



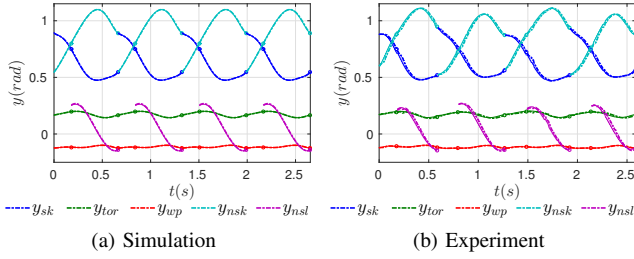


Fig. 3: Desired (straight) vs. actual (dashed) outputs: tracking performance of time dependent CLF based controllers for the simulation (left) and the experiment (right).

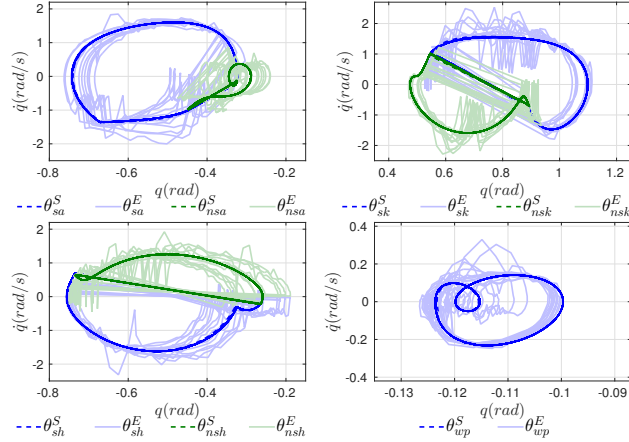


Fig. 4: Phase portraits of the ankle, knee, hip and waist pitch angles of the robot. Both simulation (S) and experimental (E) results are shown.

the time  $t$  closely matches with the phase variable  $\tau$ . This was then successfully shown in the bipedal robot DURUS. It can be observed that, similar to the notion of *Input to State Stability*, a perfect match yields zero stability, and a bounded mismatch yields boundedness to the periodic orbit. In other words, the presented controller yields a *Time or Phase to State Stable* periodic orbit which completely eliminates the determination of the phase variable for the robot. Future work involves developing this notion and extending controllers to more complex motion primitives like running and hopping.

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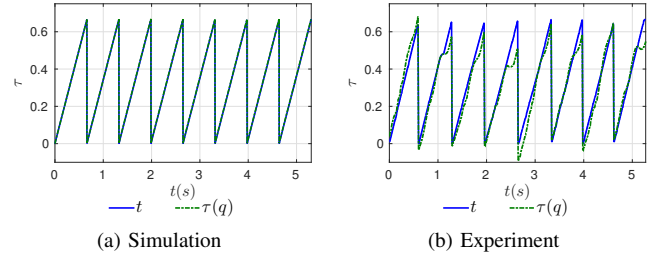


Fig. 5: Progression of time vs. the phase variable shown for simulation (left) and for the experiment (right).

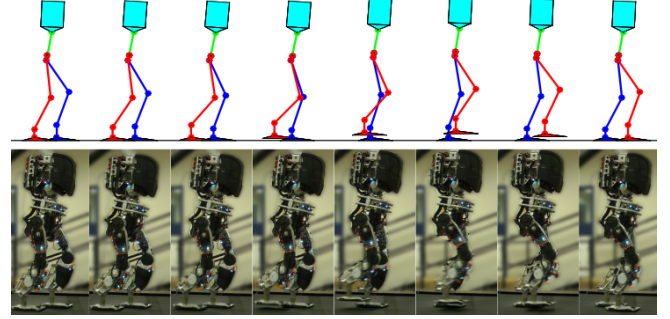


Fig. 6: Figure showing the tiles of DURUS for one step. Video link to the experiment is given in [1].

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